New Approach for Automatic Control Modeling and Analysis Using Arithmetic and Visual Fuzzy Logic-based Representations in Fully Fuzzy Environment

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Abstract - **The paper presents a new approach for the fuzzy modeling and analysis of automatic control systems in fully fuzzy environment. The approach is based on the normalized fuzzy matrices and an extension of the Arithmetic and Visual Fuzzy Logic-based Representations developed recently by Gabr and Dorrah. The approach is also suitable for fuzzy dynamical systems where all system coefficients are expressed as fuzzy parameters. It is simply based on the assignment of corresponding fuzzy levels for parameters uncertainty that is made in a heuristic way circumventing the previous difficulties in assuming probabilistic or membership functions. Implementations of the approach are carried out for solving selected automatic control problem with their parameters expressed in fully fuzzy environment. These problems involve fuzzy impulse response of systems, fuzzy Routh-Hurwitz stability criteria, fuzzy controllability and observability, and the stabilization of inverted pendulum through pole placement technique. The results demonstrated the robustness of the proposed formulation and illustrated in a systematic way how the system parameters fuzziness effect on output results can be effectively tracked for monitoring and control. Finally, it is pointed out that the suggested Arithmetic and Visual Fuzzy Logic-based Representations will open the door for a unified theory for fuzzy modeling, analysis and control for continuous and discrete automatic control systems operating in fully fuzzy environment.**

Index Terms:- Normalized Fuzzy Matrices, Arithmetic and Visual Fuzzy Logic-based Representations, Automatic Control Systems, and Inverted Pendulum System.

I. Introduction

The modeling and analysis of automatic control systems operating in fully fuzzy environment is not effectively solved in the literature [1]-[3]. There are many approaches that are carried out to handle this problem using the Conventional Fuzzy Theory. These approaches suffer many drawbacks such as with the increase of system dimensionality, the solution processing becomes very cumbersome. Moreover, the results gained by such approach are not linear and thus not reversible, leading that the results obtained in the forward path will be different than the backward path [4]-[5]. Other approaches, using the direct implementation of fuzzy matrices also has many shortcomings [6]-[8]. The main hindrance of their spread is heavily related to their implacability of its present operations (Max, Min, Max.Min, and Min.Max) as they do not reflect a corresponding real life physical meaning and causing irreversibility and nonlinearity in their processing.

Recently, Gabr and Dorrah presented their new notion of Arithmetic and Visual Fuzzy Logic-based Representations [9]-[14]. The approach is based on the normalized fuzzy matrices, where every parameter is expressed by it value and corresponding fuzzy level. It is shown by Gabr [14] that the proposed Arithmetic Fuzzy Logic-based Representation has the same mathematical function of the Conventional Fuzzy Theory. However, the new approach provides a much easier arithmetic rather than logic calculations forum that makes its application much practical and effective. Moreover, the Arithmetic Fuzzy Logic-based Representation approach possesses the key features over the Conventional Fuzzy Theory, namely; linearity, reversibility, simplicity, and applicability.

In the present work, the Arithmetic Fuzzy Logic-based Representations based on the Normalized Fuzzy Matrices is extended for presenting a new approach for fuzzy automatic control theory modeling and analysis. The new approach is based on handling systems with all their coefficients expressed in fully fuzzy environment. In the following section, a short introduction of the Arithmetic Fuzzy Logic-based Representation is firstly presented.

II. Brief Description of Arithmetic Fuzzy Logic-based Representation

The Arithmetic Fuzzy Logic-based Representation is based on expressing each parameter X by two components: X_o the deterministic equivalence, and X_f the fuzzy equivalence representing a small uncertainty or value tolerance in the parameter X [9]-[12]. The term X_f is modeled by the formula: $X_f = f_r \ell_x X_o$ where f_r is the relative unit fuzziness (usually a certain small percentage), and ℓ_x is the corresponding fuzzy level. For the sake of simplicity f_r is omitted in the representation and the parameter X is expressed by the following pair:

$$X = (X_o, \ell_x) \tag{1}$$

where the first term in the pair is the equivalent deterministic component, and ℓ_x is an integer value indicating the corresponding fuzzy level of X. The scaled or normalized fuzzy term is $\tilde{X} = f_x . \ell_x$, such that $|\tilde{X}| < 1$.

In general, the fuzzy level could be extended from level $-p$ to $+m$ where p and m are positive integers. In this case, the value of the relative unit fuzziness f_r is restricted such as $p.f_r < 1$ and $m.f_r < 1$ (preferably will be << 1). This will guarantee not changing the sign of the parameters during the formulation, and that all the *scaled* or *normalized* fuzzy components will satisfy the conditions of the normalized fuzzy matrices boundary [-1, +1]. This is not a real restriction as the whole fuzzy process is mainly based on intuitive estimation and heuristic evaluation. Similarly, we have for any other two general parameters Y and Z the following representations:

$$Y = (Y_o, \ell_y) \tag{2}$$

and

$$Z = (Z_o, \ell_z). \tag{3}$$

The algebra of Arithmetic Fuzzy Logic-based Representation was developed for the basic cases of multiplication, division, addition, and subtraction, for both *scalar* and *matrix* operations and also for function of the fuzzy-based parameters. A summary of the main fuzzy-logic based algebraic operations are presented in Table 1, following the normalized fuzzy logic-based matrices.

TABLE 1
SUMMARY OF BASIC ARITHMETIC FUZZY LOGIC-BASED REPRESENTATION ALGEBRA [9]-[14].

Name of Operation	Symbolic Representation of Operation	Resulting Values and Fuzzy Levels from Operation
Addition	X+Y	$Z_o = X_o + Y_o$, and $\ell_z = \dfrac{\ell_x X_o + \ell_y Y_o}{X_o + Y_o}$.
Subtraction	X-Y	$Z_o = X_o - Y_o$, and $\ell_z = \dfrac{\ell_x X_o - \ell_y Y_o}{X_o - Y_o}$
Multiplication	X.Y	$XY = (X_o Y_o, \ell_x + \ell_y)$·
Division	X/Y	$X/Y = (X_o/Y_o, \ell_x - \ell_y)$

The suggested approach obeys the various associative, commutative, distributive, and reverses rules. In addition, the operation sequences of the suggested technique are similar of traditional arithmetic operations avoiding using any logic operations. It was shown that the suggested approach is identical to that of the Conventional Fuzzy Theory for addition and weighted average fuzziness results for the subtraction operations. Moreover, it yields similar results of multiplications and divisions operations after ignoring the second order relative variations terms [14].

The approach is pragmatic as it requires only specifying heuristically the fuzzy logic-based levels of the parameters and coefficients, which can be relatively evaluated in real life. Moreover, for the sake of uncertainty analysis, each fuzzy level, it is assumed as an example that the parameter variation can be quantitatively modeled by a Gaussian probability function of zero mean and standard deviation σ_ℓ.

In calculating the fuzzy levels of other, the Taylor's series expansion series where used. Examples of fuzzy logic-based rules applied to some expanded functions are summarized in Table 2 where X and Y are fuzzy parameters. The selected number of the series terms will mainly depend on the accuracy required in the analysis.

TABLE 2
EXAMPLES OF CALCULATIONS OF FUNCTIONS FUZZY LEVELS USING TAYLOR'S SERIES EXPANSIONS.

Original Function	Taylor's Series Expansion	Calculated Fuzzy Level
$Y=(1+X)^{-1}$	$Y=1-X+X^2-X^3+..$	$\ell_Y = (X\ell_X + X^2.2\ell_X - X^3.3\ell_X+..)/Y$
$Y = e^{-X}$	$Y=\dfrac{1}{1+X+\dfrac{X^2}{2!}+\dfrac{X^3}{3!}}+...$	$\ell_Y = -\left(X.\ell_X + \dfrac{X^2}{2!}.2.\ell_X + \dfrac{X^3}{3!}.3.\ell_X + ...\right)/Y$
$Y = \sin X$	$Y = X - \dfrac{X^3}{3!} + \dfrac{X^5}{5!} + ...$	$\ell_Y = \left(X.\ell_X - \dfrac{X^3}{3!}.3.\ell_X + \dfrac{X^5}{5!}.5.\ell_X + ...\right)/Y$
$Y = \cos X$	$Y = 1 - \dfrac{X^2}{2!} + \dfrac{X^4}{4!} + ...$	$\ell_Y = \left(-\dfrac{X^2}{2!}.2.\ell_X + \dfrac{X^4}{4!}.4.\ell_X + ...\right)/Y$
$Y=e^{-X}\sin X$	Use expansions of both e^{-X} and $\sin X$	$\ell_Y = \{e^{-X}\} + \ell\{\sin X\}$ and sum above results
$Y=e^{-X}\cos X$	Same as above	$\ell_Y = \{e^{-X}\} + \ell\{\cos X\}$

It must be pointed out that the implementation of the Arithmetic Fuzzy Logic-based Representation to various functions strictly preserves the linearity properties. Let $X = (X_o, \ell_x)$ and $X' = (X_o, -\ell_x)$, then for any general function $F(X)$, we have $L\{F(X)\} = -L\{F(X')\}$. Similarly, all the above rules also apply to general functions of $X, Y,$ and Z, namely $F(X), G(X),$ and $H(Z)$.

III. FUZZY LOGIC-BASED AUTOMATIC CONTROL FUZZY SYSTEMS MODELING

Consider the general differential equation [1]

$$\frac{d^n y}{dt^n} + a_{n-1}.\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1.\frac{dy}{dt} + a_0.y = b_{n-1}.\frac{d^{n-1} u}{dt^{n-1}} + \ldots \\ \ldots + b_1.\frac{du}{dt} + b_0.u \tag{4}$$

where all the equation parameters are fuzzy numbers expressed by their deterministic values and corresponding fuzzy level as described by the Arithmetic Fuzzy Logic-based Representation.

Define a set of state variables for a typical fuzzy control system as follows

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \vdots \quad & \quad \vdots \end{aligned} \tag{5}$$

$$\dot{x}_n = -a_0.x_2 - \cdots - a_{n-1}.x_n + u$$

and an output equation

$$y = b'_0.x_1 + b'_1.x_2 + \cdots + b'_{n-1}.x_n \tag{6}$$

where $b'_0 \quad b'_1 \quad b'_2 \quad \cdots \quad b'_{n-1}$ are fuzzy coefficients.

Then, the state equation is expressed as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{7}$$

The state-space representation in (7) is called the controllable canonical form and the output equation is

$$y = \begin{bmatrix} b'_0 & b'_1 & b'_2 & \cdots & b'_{n-1} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} . \tag{8}$$

Consider now the state vector differential equation

$$\dot{x} = A.x + B.u . \tag{9}$$

Taking Laplace transforms of (9), we get

$$sX(s) - x(0) = A.X(s) + B.U(s) \tag{10}$$

or equivalently

$$(s.I - A).X(s) = x(0) + B.U(s) . \tag{11}$$

Using a state variable representation of a system, the characteristic equation is given by

$$\left|(s.I - A)\right| = 0 . \tag{12}$$

This yields the characteristics (closed loop form) equation [1]:

$$a_n.s^n + a_{n-1}.s^{n-1} + \ldots + a_1.s + a_0 = 0 . \tag{13}$$

The general form of the above system can be expressed in the form of system transfer function as:

$$\frac{C}{R}(s) = \frac{G(s)}{1 + G(s).H(s)} = \frac{K_c.(s - z_{c1}).(s - z_{c2})...(s - z_{cn})}{(s - p_{c1}).(s - p_{c2})...(s - p_{cn})} \tag{14}$$

where $s = p_{c1}, p_{c2}, ..., p_{cn}$ are closed-loop fuzzy poles, so called since their values make equation (11) infinite (Note that they are also the roots of the characteristic equation) and $s = z_{c1}, z_{c2}, ..., z_{cn}$ are closed-loop fuzzy zeros, since their corresponding values of (11) are zero.

We present now the handling of the general form of fuzzy control system modeling and analysis using representative examples of the fourth-order systems.

IV. Fuzzy Impulse Response of Automatic Control Transfer Function

We demonstrate in this section how a fourth orders system of the transfer function expressed by (11) can be handled in a fully fuzzy environment where all the system coefficients are expressed in the Arithmetic Fuzzy Logic-based Representation form. Let us introduce Numerical Example I that describes the fuzzy response of a high-order control system operating in fully fuzzy environment. Let us introduce this example in a general form open loop fourth-order transfer function as follows [1]:

$$X_0(s) = \frac{a_0}{s.(s + b_1).(s^2 + c_1.s + c_2)} \tag{15}$$

where a_0, b_1, c_1 and c_2 are fuzzy parameters.

Equation (15) may be written using partial fraction representation as:

$$X_0(s) = \frac{A}{s} + \frac{B}{s + b_1} + \frac{C.s + D}{(s + c_1/2)^2 + c_3^2} \tag{16}$$

where $c_3 = \sqrt{c_2 - c_1^2/4}$, A, B, C, D, and c_3 are fuzzy coefficient. Equating coefficient of (16), we get

$$\begin{aligned} (s^3): &\quad 0 = A + B + C \\ (s^2): &\quad 0 = A.(b_1 + c_1) + B.c_1 + C.b_1 \\ (s^1): &\quad 0 = A.(c_2 + b_1.c_1) + B.c_2 + D.b_1 \\ (s^0): &\quad a_0 = A.b_1.c_2 \end{aligned} \tag{17}$$

The fuzzy levels of the various parameters are calculated using various rules of the Arithmetic Fuzzy Logic-based approach. As an example, we have:

$$\begin{aligned} \ell_{b_1+c_1} &= \ell\{b_1 + c_1\} \\ &= \frac{b_1.\ell_{b_1} + c_1.\ell_{c_1}}{b_1 + c_1} \end{aligned} \tag{18}$$

$$\begin{aligned} \ell_{b_1.c_1} &= \ell\{b_1.c_1\} \\ &= \ell_{b_1} + \ell_{c_1} \end{aligned} \tag{19}$$

$$\begin{aligned} \ell_{c_2+b_1.c_1} &= \ell\{c_2 + b_1.c_1\} \\ &= \frac{c_2.\ell_{c_2} + b_1.c_1.(\ell_{b_1} + \ell_{c_1})}{c_2 + b_1.c_2} . \end{aligned} \tag{20}$$

Using the Gaussian Elimination technique, the above matrix equation can be solved with its corresponding fuzzy levels.

Numerical Example I:

As a numerical example, we choose the value of fuzzy parameters as shown in Table 3. The results of parameters $A, B, C,$ and D are also shown in Table 3. Accordingly, the inverse transform of (16), may be expressed as:

$$X_0(t) = A - B.e^{b_1.t} - C.e^{c_1.t/2}[(c_4)^2.\sin c_3.t - \cos c_3.t] \quad (21)$$

such that $C_4 = \left(\frac{D}{C} - \frac{c_1}{2}\right)$ where A, B, C, D, b_1, c_1, c_3, and C_4 are fuzzy parameters. For the selected scenarios shown in Table 3, the fuzzy levels of their impulse responses are given in Table 5 and sketched in Figure 1.

TABLE 3
DESCRIPTION OF FUZZY INPUT AND OUTPUT DATA FOR SELECTED SCENARIOS OF NUMERICAL EXAMPLE I.

Parameter	Value	Corresponding Fuzzy Level Values			
		Scenario I	Scenario II	Scenario III	Scenario IV
a_0	12.5	3	1	-1	-3
b_1	0.5	-3	-1	1	3
c_1	1.0	-2	-2	2	2
c_2	25.0	3	1	-1	-3
A	1.000	3	1	-1	-3
B	-1.010	3	1	-1	-3
C	0.010	-4	-4	4	4
D	-0.495	0	0	0	0

The equations were solved in Excel sheet with built-in functions programmed using Visual Basic Applications (VBAs). It follows from the sketches of the impulse time response of Figure 1 and Table 4 that the fuzziness is related to the time instant. The color of the response is an indication of the fuzzy level using the color system shown in Table 5.

The colors are selected arbitrary without restricting that each corresponding positive and negative colors are conjugates (summation is either white or black). This is equivalent to the Visual Fuzzy Logic-based Representation presented by Gabr and Dorrah [9]-[12]. The approach is robust and produces a more in-depth heuristic values to the fuzziness of the variables that can be further transferred to corresponding uncertainties in these variables by assuming value of the relative fuzziness f and the standard deviation σ of its variability.

TABLE 4
SYSTEM IMPULSE RESPONSE AND CORRESPONDING FUZZY LEVELS FOR SELECTED SCENARIOS OF NUMERICAL EXAMPLE I.

Ser.	Time (Sec.)	$X_0(t)$	Fuzzy Level Values			
			Scenario I	Scenario II	Scenario III	Scenario IV
1	0.0	0.00000	0	0	0	0
2	1.0	0.38820	1	0	0	-1
3	2.0	0.62573	1	0	0	-1
4	3.0	0.77107	2	1	-1	-2
5	4.0	0.85999	2	1	-1	-2
6	5.0	0.91440	3	1	-1	-3
7	6.0	0.94768	3	1	-1	-3
8	7.0	0.96803	3	1	-1	-3
9	8.0	0.98047	3	1	-1	-3
10	9.0	0.98808	3	1	-1	-3
11	10.0	0.99273	3	1	-1	-3

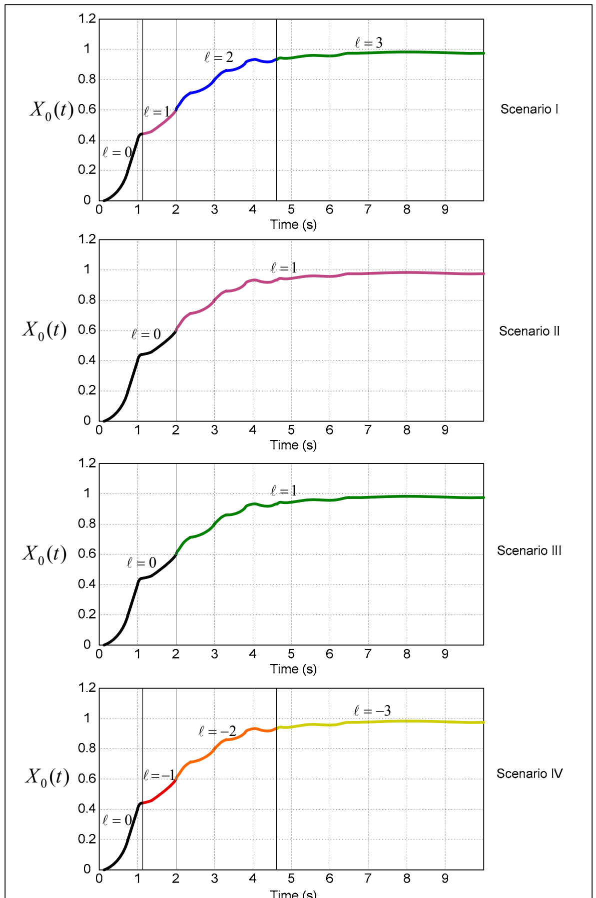

Figure 1 Fuzzy Impulse Response using Visual Representation of Example I.

TABLE 5
DEFINITION OF POSITIVE AND NEGATIVE COLORS SAMPLE SCALE FOR VISUAL FUZZY REPRESENTATION.

Ser.	Color	Color Code	RBG Color Index	Excel Color Index	Fuzzy Level	Type
1		Green Light	(204,255,204)	35	+6	Positive Colors
2		Blue Light	(153,204,255)	37	+5	
3		Violent Light	(204,153,255)	39	+4	
4		Green	(51,153,102)	50	+3	
5		Blue	(0,102,204)	5	+2	
6		Violent (Lavender)	(255,0,255)	7	+1	

Ser.	Color	Color Code	RBG Color Index	Excel Color Index	Fuzzy Level	Type
7		Black	(0,0,0)	2	0	0
8		Red	(255,0,0)	3	-1	Negative Colors
9		Orange	(255,102,0)	46	-2	
10		Yellow	(255,255,0)	27	-3	
11		Red Light	(255,153,204)	38	-4	
12		Orange Light	(255,153,0)	45	-5	
13		Yellow Light	(255,255,204)	19	-6	

The results reveal the strict linearity of the proposed fuzzy logic-based approach. For, the input fuzzy levels of scenarios I and IV, and II, and III selected identical with opposite signs, the corresponding output results are also identical with opposite sings.

V. Routh-Hurwitz Criterion for Fuzzy Control System Stability

The work of Routh and Hurwitz [1] gives a method of indicating the presence and number of unstable roots, but not their value. Consider the general form of characteristic equation expressed by (13). The Routh-Hurwitz stability criterion states that: For there to be no roots with positive real parts then there is a necessary, but not sufficient, condition that all coefficients in the characteristic equation have the same sign and that none are zero. If the above is satisfied, then the necessary and sufficient condition for stability is either:

a) All the Hurwitz determinants of the polynomial are positive, or alternatively
b) All coefficients of the first column of Routh array have the same sign. The number of sign changes indicates the number of unstable roots.

The Hurwitz determinants are [1]:

$$D_1 = a_1 \qquad D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \qquad D_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 0 & a_2 & a_4 \end{vmatrix} \tag{22}$$

…etc.

such that all parameters are expressed in the Arithmetic Fuzzy Logic-based form. All the above determinant operations are carried out following the Arithmetic Fuzzy Logic-based operation expressed in Table 1.

Numerical Example II:

Let us check the stability of the closed loop control system of Numerical Example I, where the open transfer function is expressed in (15), and is having a unity feedback. The closed loop characteristic function can be expressed as:

$$s.(s+b_1).(s^2+c_1.s+c_2)+a_0=0 \tag{23}$$

or equivalently

$$s^4+(c_1+b_1).s^3+(c_2+c_1.b_1).s^2+c_2.b_1.s+a_0=0 \tag{24}$$

where a_0, b_1, c_1 and c_2 are fuzzy parameters following the scenarios given in Table 3. Applying now the Routh-Hurwitz Criterion, we have first test all the coefficients are present and have the same sign. The second test is to check the fuzzy determinates D_1, D_2, and D_3 for different scenarios.

These Hurwitz determinants for this numerical example can be expressed as:

$$D_1 = c_2.b_1 \tag{25}$$

$$D_2 = \begin{vmatrix} c_2.b_1 & c_1+b_1 \\ a_1 & c_2+c_1.b_1 \end{vmatrix} \tag{26}$$

and

$$D_3 = \begin{vmatrix} c_2.b_1 & c_1+b_1 & 0 \\ a_1 & c_2+c_1.b_1 & 1 \\ 0 & c_2.b_1 & c_1+b_1 \end{vmatrix}. \tag{27}$$

The numerical results are shown in Table 6.

TABLE 6
RESULTS OF ROUTH-HURWITZ FUZZY DETERMINANTS OF NUMERICAL EXAMPLE II.

Ser.	Aspect	Value	Fuzzy Level: Scenario I	Scenario II	Scenario III	Scenario IV
1	$\lvert D_1 \rvert$	12.5	3	1	-1	-3
2	$\lvert D_2 \rvert$	300.0	3	1	-1	-3
3	$\lvert D_3 \rvert$	443.75	1	-1	1	-1

The Hurwitz determinants of the polynomial are all positive with various level of fuzziness. The fuzzy levels of determinants describe the level of uncertainly in the results. For different scenarios, we have same fuzzy results with opposite sign of the fuzziness of the determinant of scenarios with opposite input fuzzy data indicating the linearly of the Arithmetic Fuzzy Logic-based Approach.

VI. Controllability and Observability for Fuzzy Logic-based Control Systems

A system is said to be controllable if a control vector $u(t)$ exists that will transfer the system from any initial state $x(t_0)$ to some final state $x(t)$ in a finite time interval.

A system is said to be observable if at time t_0, the system state $x(t_0)$ can be exactly determined from observation of the output $y(t)$ over a finite time interval.

If system is described by

$$\dot{x} = A.x + B.u$$
$$y = C.x + D.u \tag{28}$$

then a sufficient condition for complete state controllability is that the n x n matrix [1]:

$$M = [B : A.B : \ldots : A^{n-1}.B] \tag{29}$$

contains n linearly independent row or column vectors, i.e. is of rank n (that is, the matrix is non-singular, i.e. the determinant is non-zero). Equation (29) is called the controllability matrix.

The system described by (29) is completely observable if the n x n matrix [1]:

$$N = [C^T : A^T.C^T : \cdots : (A^T)^{n-1}.C^T] \tag{30}$$

where all the system coefficients are expressed in the Arithmetic Fuzzy Logic-based Representation form.

Numerical Example III:

Consider the state space representation of the closed system of the Numerical Example I:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{bmatrix} u \tag{31}$$

and

$$y = [\beta \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{32}$$

such that $a_1 = c_2.b_1$, $a_2 = c_2 + c_1.b_1$, $a_3 = c_1 + b_1$ and α, β are fuzzy parameters, where $\alpha = (1,1)$ and $\beta = (1,-1)$ (Case 1).

Now let us form the controllability matrix

$$M = [B : A.B : A^2.B : A^3.B]$$
$$= \alpha^4 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -a_3 \\ 0 & 1 & -a_3 & -a_2 + a^2 \\ 1 & -a_3 & -a_2 + a^2 & -a_0 + 2.a_2.a_3 - a_3^2 \end{bmatrix} \tag{33}$$

Determinant of $M = \alpha^4 \neq 0$, with fuzzy level $= 4.\ell_\alpha$. Thus the system is controllable. The fuzzy levels of various scenarios are illustrated in Table 7. The results indicate that for this example the fuzzy level of the controllability matrix determinant is a function only of α^4 and not related to other system parameters fuzziness.

Applying the Observability matrix criterion for Case 1, we have

$$N = [C^T : A^T.C^T : (A^T)^2.C^T : (A^T)^3.C^T] = \beta^4.I \tag{34}$$

where I is the identity matrix. The determinant of $N = \beta^4 \neq 0$, with fuzzy level $= 4.\ell_\beta$, thus the system is observable. The associated fuzzy levels of various scenarios are given in Table 8. It follows from the example that the fuzzy level of the Observability matrix determinant for this case is a function of β^4 and is not related to other system parameter fuzziness.

Now let us examine Case 2, where

$$y = [0 \quad 0 \quad 0 \quad \beta] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{35}$$

such that $\beta = (1,0)$. Applying the Observability matrix criterion for Case 2, we have

$$N = [C^T : A^T.C^T : (A^T)^2.C^T : (A^T)^3.C^T]$$
$$= \beta^4 \begin{bmatrix} 0 & -a_0 & a_0 a_3 & a_0.a_2 - a_0.a_3^2 \\ 0 & -a_1 & -a_0 + a_0.a_3 & a_0.a_3 + a_2.a_3 + a_1.a_3^2 \\ 0 & -a_2 & -a_1 + a_2.a_3 & -a_0 + a_0.a_3 \\ 1 & -a_3 & -a_2 + a_3^2 & -a_0 + 2a_2.a_3 + a_3^2 \end{bmatrix} \tag{36}$$

The determinant of $N \neq 0$, thus the system is observable. The associated fuzzy levels of various scenarios are given in Table 7. It follows from the example that the fuzzy level of the Observability matrix determinant is a function of β^4 and also related to the system parameter fuzziness. The results indicate a relatively high fuzziness of the Observability Matrix due to the fuzziness of the system's parameters.

TABLE 7
RESULTS OF FUZZY CONTROLLABILITY AND OBSERVABILITY MATRIX CRITERIA OF NUMERICAL EXAMPLE III.

Ser.	Aspect	Value	Fuzzy Level			
			Scenario I	Scenario II	Scenario III	Scenario IV
1	$\|M\|$	1	4	4	4	4
2	$\|N\|$: Case 1	2.44x10^4	-4	-4	-4	-4
3	$\|N\|$: Case 2	-58085.9	4	-2	2	-4

VII. MODELING AND STABILIZATION OF INVERTED PENDULUM OPERATING IN FULLY FUZZY ENVIRONMENT

In this section, the suggested approach is implemented for the fuzzy modeling and stabilization of the inverted pendulum system (*Numerical Example IV*) as shown in Figure 2. The inverted pendulum problem is a classic example of producing a stable closed-loop control system from an unstable plant. Since the system can be modeled, it is possible to design a controller using the pole placement techniques [1].

In the figure, m is the mass, L denotes the half-length of the pendulum and M' is the mass of the trolley. $F(t)$ indicates the applied force to the trolley in the x-direction.

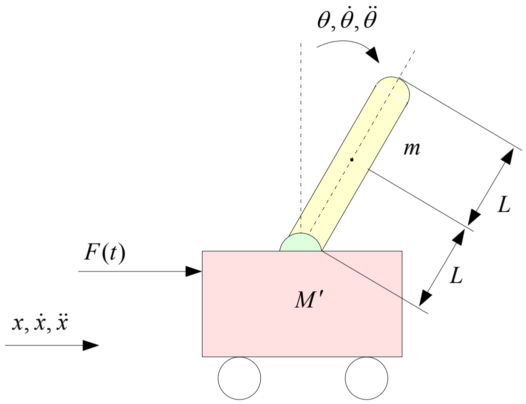

Figure 2 Sketch Showing Main Parameters of the Inverted Pendulum (Numerical Example IV) [1].

It is assumed that θ is small and second-order terms $(\dot{\theta}^2)$ can be neglected, then we can define the state variables of the inverted pendulum system as ($g = 9.81$):

$$x_1 = \theta, \quad x_2 = \dot{\theta}, \quad x_3 = x \quad \text{and } x_4 = \dot{x} \tag{37}$$

and the control variable is

$$u = F(t) \tag{38}$$

and from (37) and (38), the state equations become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_{41} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix} u \tag{39}$$

where

$$\begin{aligned} a_{21} &= \frac{3.g.(M'+m)}{L.\{4.(M'+m)-3.m\}} \\ a_{41} &= \frac{-3g.m}{4.(M'+m)-3.m} \\ b_2 &= \frac{-3}{L.\{4.(M'+m)-3.m\}} \\ b_4 &= \left(\frac{1}{M'+m}\right).\left\{1+\frac{3.m}{4.(M'+m)-3.m}\right\} \end{aligned} \tag{40}$$

The Controllability matrix M can be expressed as:

$$M = \begin{bmatrix} 0 & b_2 & 0 & a_{21}.b_2 \\ b_2 & 0 & a_{21}.b_2 & 0 \\ 0 & b_4 & 0 & a_{41}.b_2 \\ b_4 & 0 & a_{41}.b_2 & 0 \end{bmatrix}. \tag{41}$$

Data for simulation are represented in Table 8 for different selected scenarios. The output equation is

$$y = C.x \tag{42}$$

where C is the identity matrix. For a regulator, with a scalar control variable

$$u = -K.x\,. \tag{43}$$

TABLE 8
DESCRIPTION OF INPUT AND OUTPUT DATA FOR SELECTED SCENARIOS OF NUMERICAL EXAMPLE IV.

Parameter	Value	Corresponding Fuzzy Level Values			
		Scenario I	Scenario II	Scenario III	Scenario IV
L	1	6	4	-4	-6
M'	1	-4	-2	2	4
m	0.5	6	4	-4	-6
a_{21}	9.81	-4	-3	3	4
a_{41}	-3.27	-3	-3	3	3
b_2	-0.6667	-3	-3	3	3
b_4	0.8889	3	1	-1	-3
$\lvert M \rvert$	19.0	-5	-6	6	5
k_1	-174.907	-1	0	0	1
k_2	-57.144	-1	0	0	1
k_3	-39.144	1	1	-1	-1
k_4	-29.358	2	2	-2	-2

The elements of K can be obtained by selecting a set of desired closed-loop poles by the Ackermann's formula [1] for system stabilization through the pole placement technique. Let

$$K = [0 \quad 0 \quad 0 \quad \cdots \quad 1].M^{-1}.\phi(A) \tag{44}$$

where M is the controllability matrix and

$$\phi(A) = A^n + \alpha_{n-1}.A^{n-1} + \cdots + \alpha_1.A + \alpha_0.I \tag{45}$$

where A is the system matrix and α_i are the coefficients of the desired closed-loop characteristic equation.

If the required closed-loop poles are $s = -2 \pm j2$ for the pendulum, and $s = -4 \pm j4$ for the trolley, then the closed-loop determination characteristic equation becomes:

$$s^4 + 12s^3 + 72s^2 + 192s + 256 = 0\,. \tag{46}$$

The algebraic form of the fuzzy gain result can be expressed as follows; let

$$[\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4] = [0 \quad 0 \quad 0 \quad 1]M^{-1} \tag{47}$$

then we can attain by simple matrix operation the following fuzzy values of the gain vector K,

$$k_1 = \beta_1.\left(\alpha_o + \alpha_2.a_{21} + \alpha_4.a_{21}^2\right) + \beta_2\left(\alpha_2.a_{41} + \alpha_4.a_{21}.a_{41}\right) \tag{48}$$

$$k_2 = \beta_1.(\alpha_1 + \alpha_3.a_{21}) + \beta_3.\alpha_3.a_{41} \tag{49}$$

$$k_3 = \beta_3.\alpha_0 \tag{50}$$

and

$$k_4 = \beta_3.\alpha_1 . \quad (51)$$

Applying the Arithmetic Fuzzy Logic-based operations to the numeric value above equation, we arrive at the results of fuzzy gain vector as shown in Table 8. The final fuzzy results of the output feedback gain demonstrate the final effect of input fuzziness on the output fuzziness, and confirm the robustness of the suggested technique. Moreover, the various testing showed that the most effective fuzziness is that of the mass of the trolley M'. Finally, efforts should be concentrated on building systems with parameters such that their output feedback gains satisfy the required fuzzy levels.

VIII. Conclusions

A new approach for the fuzzy control systems' modeling and analysis using the Arithmetic and Visual Fuzzy Logic-based Representation technique was presented. Key implementations issues in fuzzy control systems were addressed in a very smooth and systematic way. These issues covered system's fuzzy impulse response, system's stability using Routh-Hurwitz Criteria, system's controllability and observability, and the stabilization of inverted pendulum through the pole placement technique. Numerical examples of fourth-order systems were solved to demonstrate the effectiveness and applicability of the new approach. As the approach is mainly based on normalized fuzzy matrices, the new approach is open for high dimensional systems.

The applications showed the high robustness of the technique and its ability for tracking the system fuzziness all over the steps of the solution. Moreover, the suggested approach possesses very key properties compared to the Conventional Fuzzy Theory; these are the linearity and reversibility. This means the fuzziness results are identical for treatment of forward paths or through backward paths

The suggested approach will open the door for more general modeling and analysis of both *continuous* and *discrete* control systems. Moreover, it provides an effective tool for fuzzy levels tracking during problems' solutions. Examples of other foreseen extensions are as follows [1]-[3]:

i) Design of Fuzzy P, PI, PD, and PID control systems.
ii) Digital and discrete system fuzzy modeling and analysis.
iii) State-space methods for fuzzy control system analysis and design.
iv) Design of full and reduced orders fuzzy state observers and estimators for closed-loop systems.
v) Optimal and robust fuzzy control of multivariate systems.
vi) Other problems such as multivariate fuzzy Kalman state estimation, fuzzy linear quadratic regulators, and fuzzy Lyapunov Stability criterion.

Finally, it is recommended that the suggested approach be further developed for building the ***Global Fuzzy Theory*** or ***Theory of Global Fuzziness*** to be applied for analyzing the internal consolidity of real life systems operating in fully fuzzy environment. In general, a good consolidated (tied up or well attached together) system will have corresponding low overall output levels of fuzziness for high combined input plus parameters fuzziness, and vice versa for unconsolidated systems. Based on that, system's internal behavior can be effectively judged for its degree *of internal systems' consolidity* against any intruded inside and outside fuzziness. This will represent new internal information (feature) of systems.

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